Part IV
Recursion

By now you're very familiar with the idea of implementing a function by composing other functions. In effect we are breaking down a large problem into smaller parts. The idea of recursion—as usual, it sounds simpler than it actually is—is that one of the smaller parts can be the *same* function we are trying to implement.

At clothes stores they have arrangements with three mirrors hinged together. If you keep the side mirrors pointing outward, and you're standing in the right position, what you see is just three separate images of yourself, one face-on and two with profile views. But if you turn the mirrors in toward each other, all of a sudden you see what looks like infinitely many images of yourself. That's because each mirror reflects a scene that includes an image of the mirror itself. This *self-reference* gives rise to the multiple images.

Recursion is the idea of self-reference applied to computer programs. Here's an example:

> "I'm thinking of a number between 1 and 20."
>
> (Her number is between 1 and 20. I'll guess the halfway point.) "10."
>
> "Too low."
>
> (Hmm, her number is between 11 and 20. I'll guess the halfway point.) "15."
>
> "Too high."
>
> (That means her number is between 11 and 14. I'll guess the halfway point.) "12."
>
> "Got it!"

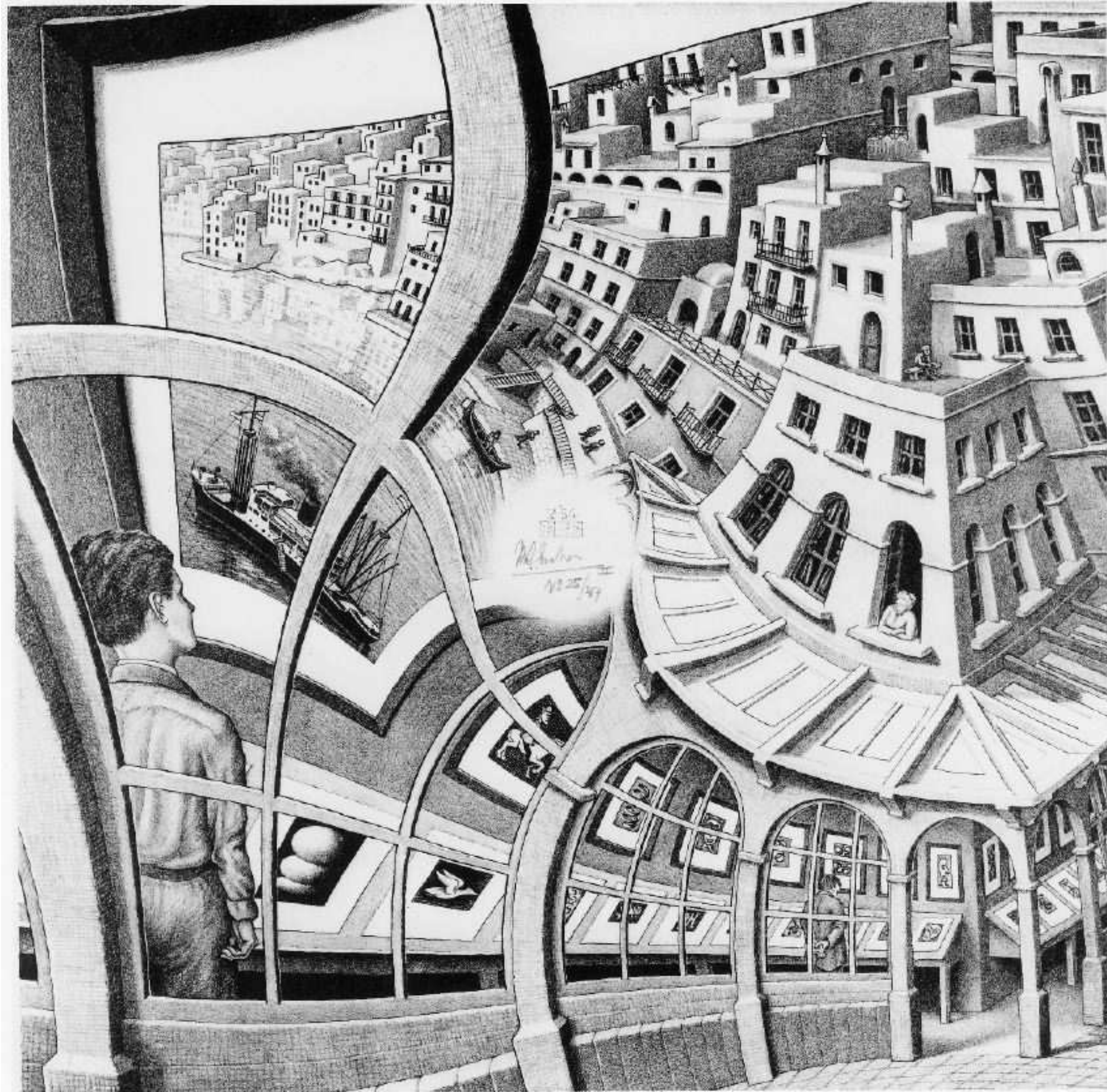
We can write a procedure to do this:

```
(define (game low high)
  (let ((guess (average low high)))
    (cond ((too-low? guess) (game (+ guess 1) high))
          ((too-high? guess) (game low (- guess 1)))
          (else '(I win!)))))
```

This isn't a complete program because we haven't written `too-low?` and `too-high?`. But it illustrates the idea of a problem that contains a version of itself as a subproblem: We're asked to find a secret number within a given range. We make a guess, and if it's not the answer, we use that guess to create another problem in which the same secret number is known to be within a smaller range. The self-reference of the problem description is expressed in Scheme by a procedure that invokes itself as a subprocedure.

Actually, this isn't the first time we've seen self-reference in this book. We defined "expression" in Chapter 3 self-referentially: An expression is either atomic or composed of smaller expressions.

The idea of self-reference also comes up in some paradoxes: Is the sentence "This sentence is false" true or false? (If it's true, then it must also be false, since it says so; if it's false, then it must also be true, since that's the opposite of what it says.) This idea also appears in the self-referential shapes called *fractals* that are used to produce realistic-looking waves, clouds, mountains, and coastlines in computer-generated graphics.



Print Gallery, by M. C. Escher (lithograph, 1956)

11 Introduction to Recursion

I know an old lady who swallowed a fly.
I don't know why she swallowed the fly.
Perhaps she'll die.

I know an old lady who swallowed a spider
that wriggled and jiggled and tickled inside her.
She swallowed the spider to catch the fly.
I don't know why she swallowed the fly.
Perhaps she'll die.

I know an old lady who swallowed a bird.
How absurd, to swallow a bird!
She swallowed the bird to catch the spider
that wriggled and jiggled and tickled inside her.
She swallowed the spider to catch the fly.
I don't know why she swallowed the fly.
Perhaps she'll die.

I know an old lady who swallowed a cat.
Imagine that, to swallow a cat.
She swallowed the cat to catch the bird.
She swallowed the bird to catch the spider
that wriggled and jiggled and tickled inside her.
She swallowed the spider to catch the fly.
I don't know why she swallowed the fly.
Perhaps she'll die.

I know an old lady who swallowed a dog.
What a hog, to swallow a dog!
She swallowed the dog to catch the cat.
She swallowed the cat to catch the bird.
She swallowed the bird to catch the spider
that wriggled and jiggled and tickled inside her.
She swallowed the spider to catch the fly.
I don't know why she swallowed the fly.
Perhaps she'll die.

I know an old lady who swallowed a horse.
She's dead of course!

100 bottles of beer on the wall,
100 bottles of beer.
If one of those bottles should happen to fall,
99 bottles of beer on the wall.

99 bottles of beer on the wall,
99 bottles of beer.
If one of those bottles should happen to fall,
98 bottles of beer on the wall.

98 bottles of beer on the wall,
98 bottles of beer.
If one of those bottles should happen to fall,
97 bottles of beer on the wall.

97 bottles of beer on the wall,
97 bottles of beer.
If one of those bottles should happen to fall,
96 bottles of beer on the wall. . .

In the next few chapters we're going to talk about *recursion:* solving a big problem by reducing it to a similar, smaller subproblem. Actually that's a little backward from the old lady in the song, who turned her little problem into a similar but *bigger* problem! As the song warns us, this can be fatal.

Here's the first problem we'll solve. We want a function that works like this:

```
> (downup 'ringo)
(RINGO RING RIN RI R RI RIN RING RINGO)

> (downup 'marsupial)
(MARSUPIAL
 MARSUPIA
 MARSUPI
 MARSUP
 MARSU
 MARS
 MAR
 MA
 M
 MA
 MAR
 MARS
 MARSU
 MARSUP
 MARSUPI
 MARSUPIA
 MARSUPIAL)
```

None of the tools that we've used so far will handle this problem. It's not a "compute this function for each letter of the word" problem, for which we could use `every`.* Rather, we have to think about the entire word in a rather complicated way.

We're going to solve this problem using recursion. It turns out that the idea of recursion is both very powerful—we can solve a *lot* of problems using it—and rather tricky to understand. That's why we're going to explain recursion several different ways in the coming chapters. Even after you understand one of them, you'll probably find that thinking about recursion from another point of view enriches your ability to use this idea. The explanation in this chapter is based on the *combining method.*

---

* If your instructor has asked you to read Part IV before Part III, ignore that sentence.

A Separate Procedure for Each Length

Since we don't yet know how to solve the `downup` problem in general, let's start with a particular case that we *can* solve. We'll write a version of `downup` that works only for one-letter words:

```
(define (downup1 wd)
  (se wd))

> (downup1 'a)
(A)
```

So far so good! This isn't a very versatile program, but it does have the advantage of being easy to write.

Now let's see if we can do two-letter words:

```
(define (downup2 wd)
  (se wd (first wd) wd))

> (downup2 'be)
(BE B BE)
```

Moving right along...

```
(define (downup3 wd)
  (se wd
      (bl wd)
      (first wd)
      (bl wd)
      wd))

> (downup3 'foo)
(FOO FO F FO FOO)
```

We could continue along these lines, writing procedures `downup4` and so on. If we knew that the longest word in English had 83 letters, we could write all of the single-length `downup`s up to `downup83`, and then write one overall `downup` procedure that would consist of an enormous `cond` with 83 clauses, one for each length.

Use What You Have to Get What You Need

But that's a terrible idea. We'd get really bored, and start making a lot of mistakes, if we tried to work up to downup83 this way.

The next trick is to notice that the middle part of what (downup3 'foo) does is just like (downup2 'fo):

```
> (downup3 'foo)
(FOO  FO F FO  FOO)
        ↑
   (downup2 'fo)
```

So we can find the parts of downup3 that are responsible for those three words:

```
(define (downup3 wd)
  (se wd
      (bl wd)
      (first wd)
      (bl wd)
      wd))

            (FOO  FO F FO  FOO)
```

and replace them with an invocation of downup2:

```
(define (downup3 wd)
  (se wd (downup2 (bl wd)) wd))
```

How about downup4? Once we've had this great idea about using downup2 to help with downup3, it's not hard to continue the pattern:

```
(define (downup4 wd)
  (se wd (downup3 (bl wd)) wd))

> (downup4 'paul)
(PAUL PAU PA P PA PAU PAUL)
```

The reason we can fit the body of downup4 on one line is that most of its work is done for it by downup3. If we continued writing each new downup procedure independently, as we did in our first attempt at downup3, our procedures would be getting longer and longer. But this new way avoids that.

```
(define (downup59 wd)
  (se wd (downup58 (bl wd)) wd))
```

Also, although it may be harder to notice, we can even rewrite `downup2` along the same lines:

```
(define (downup2 wd)
  (se wd (downup1 (bl wd)) wd))
```

Notice That They're All the Same

Although `downup59` was easy to write, the problem is that it won't work unless we also define `downup58`, which in turn depends on `downup57`, and so on. This is a lot of repetitive, duplicated, and redundant typing. Since these procedures are all basically the same, what we'd like to do is combine them into a single procedure:

```
(define (downup wd)                          ;; first version
  (se wd (downup (bl wd)) wd))
```

Isn't this a great idea? We've written one short procedure that serves as a kind of abbreviation for 59 other ones.

Notice That They're Almost All the Same

Unfortunately, it doesn't work.

```
> (downup 'toe)
ERROR: Invalid argument to BUTLAST: ""
```

What's gone wrong here? Not quite every numbered `downup` looks like

```
(define (downupn wd)
  (se wd (downupn-1 (bl wd)) wd))
```

The only numbered `downup` that doesn't follow the pattern is `downup1`:

```
(define (downup1 wd)
  (se wd))
```

So if we collapse all the numbered downups into a single procedure, we have to treat one-letter words as a special case:

```
(define (downup wd)
  (if (= (count wd) 1)
      (se wd)
      (se wd (downup (bl wd)) wd)))

> (downup 'toe)
(TOE TO T TO TOE)

> (downup 'banana)
(BANANA BANAN BANA BAN BA B BA BAN BANA BANAN BANANA)
```

This version of downup will work for any length word, from a to pneumonoultramicroscopicsilicovolcanoconinosis* or beyond.

Base Cases and Recursive Calls

Downup illustrates the structure of every recursive procedure. There is a choice among expressions to evaluate: At least one is a *recursive* case, in which the procedure (e.g., downup) itself is invoked with a smaller argument; at least one is a *base* case, that is, one that can be solved without calling the procedure recursively. For downup, the base case is a single-letter argument.

How can this possibly work? We're defining downup in terms of downup. In English class, if the teacher asks you to define "around," you'd better not say, "You know, *around!*" But we appear to be doing just that. We're telling Scheme: "In order to find downup of a word, find downup of another word."

The secret is that it's not just any old other word. The new word is *smaller* than the word we were originally asked to downup. So we're saying, "In order to find downup of a word, find downup of a shorter word." We are posing a whole slew of *subproblems* asking for the downup of words smaller than the one we started with. So if someone asks us the downup of happy, along the way we have to compute the downups of happ, hap, ha, and h.

* It's a disease. Coal miners get it.

A recursive procedure doesn't work unless every possible argument can eventually be reduced to some base case. When we are asked for `downup` of `h`, the procedure just knows what to do without calling itself recursively.

We've just said that there has to be a base case. It's also important that each recursive call has to get us somehow closer to the base case. For `downup`, "closer" means that in the recursive call we use a shorter word. If we were computing a numeric function, the base case might be an argument of zero, and the recursive calls would use smaller numbers.

Pig Latin

Let's take another example; we'll write the Pig Latin procedure that we showed off in Chapter 1. We're trying to take a word, move all the initial consonants to the end, and add "ay."

The simplest case is that there are no initial consonants to move:

```
(define (pigl0 wd)
  (word wd 'ay))

> (pigl0 'alabaster)
ALABASTERAY
```

(This will turn out to be the base case of our eventual recursive procedure.)

The next-simplest case is a word that starts with one consonant. The obvious way to write this is

```
(define (pigl1 wd)                              ;; obvious version
  (word (bf wd) (first wd) 'ay))

> (pigl1 'salami)
ALAMISAY
```

but, as in the `downup` example, we'd like to find a way to use `pigl0` in implementing `pigl1`. This case isn't exactly like `downup`, because there isn't a piece of the return value that we can draw a box around to indicate that `pigl0` returns that piece. Instead, `pigl0` puts the letters `ay` at the end of some word, and so does `pigl1`. The difference is that `pigl1` puts `ay` at the end of a *rearrangement* of its argument word. To make this point clearer, we'll rewrite `pigl1` in a way that separates the rearrangement from the `ay` addition:

```
(define (pigl1 wd)
  (word (word (bf wd) (first wd))
        'ay))

> (pigl1 'pastrami)
ASTRAMIPAY
```

Now we actually replace the `pigl0`-like part with an invocation. We want to replace `(word` *`something`* `'ay)` with `(pigl0` *`something`*`)`. If we use `pigl0` to attach the `ay` at the end, our new version of `pigl1` looks like this:

```
(define (pigl1 wd)
  (pigl0 (word (bf wd) (first wd))))
```

How about a word starting with two consonants? By now we know that we're going to try to use `pigl1` as a helper procedure, so let's skip writing `pigl2` the long way. We can just move the first consonant to the end of the word, and handle the result, a word with only one consonant in front, with `pigl1`:

```
(define (pigl2 wd)
  (pigl1 (word (bf wd) (first wd))))

> (pigl2 'trample)
AMPLETRAY
```

For a three-initial-consonant word we move one letter to the end and call `pigl2`:

```
(define (pigl3 wd)
  (pigl2 (word (bf wd) (first wd))))

> (pigl3 'chrome)
OMECHRAY
```

So how about a version that will work for any word?* The recursive case will involve

* As it happens, there are no English words that start with more than four consonants. (There are only a few even with four; "phthalate" is one, and some others are people's names, such as "Schneider.") So we could solve the problem without recursion by writing the specific procedures up to `pigl4` and then writing a five-way `cond` to choose the appropriate specific case. But as you will see, it's easier to solve the more general case! A single recursive procedure, which can handle even nonexistent words with hundreds of initial consonants, is less effort than the conceptually simpler four-consonant version.

taking the `pigl` of `(word (bf wd) (first wd))`, to match the pattern we found in `pigl1`, `pigl2`, and `pigl3`. The base case will be a word that begins with a vowel, for which we'll just add `ay` on the end, as `pigl0` does:

```
(define (pigl wd)
  (if (member? (first wd) 'aeiou)
      (word wd 'ay)
      (pigl (word (bf wd) (first wd)))))
```

It's an unusual sense in which `pigl`'s recursive call poses a "smaller" subproblem. If we're asked for the `pigl` of `scheme`, we construct a new word, `chemes`, and ask for `pigl` of that. This doesn't seem like much progress. We were asked to translate `scheme`, a six-letter word, into Pig Latin, and in order to do this we need to translate `chemes`, another six-letter word, into Pig Latin.

But actually this *is* progress, because for Pig Latin the base case isn't a one-letter word, but rather a word that starts with a vowel. `Scheme` has three consonants before the first vowel; `chemes` has only two consonants before the first vowel.

`Chemes` doesn't begin with a vowel either, so we construct the word `hemesc` and try to `pigl` that. In order to find `(pigl 'hemesc)` we need to know `(pigl 'emesch)`. Since `emesch` *does* begin with a vowel, `pigl` returns `emeschay`. Once we know `(pigl 'emesch)`, we've thereby found the answer to our original question.

Problems for You to Try

You've now seen two examples of recursive procedures that we developed using the combining method. We started by writing special-case procedures to handle small problems of a particular size, then simplified the larger versions by using smaller versions as helper procedures. Finally we combined all the nearly identical individual versions into a single recursive procedure, taking care to handle the base case separately.

Here are a couple of problems that can be solved with recursive procedures. Try them yourself before reading further. Then we'll show you our solutions.

```
> (explode 'dynamite)
(D Y N A M I T E)

> (letter-pairs 'george)
(GE EO OR RG GE)
```

Our Solutions

What's the smallest word we can `explode`? There's no reason we can't explode an empty word:

```
(define (explode0 wd)
  '())
```

That wasn't very interesting, though. It doesn't suggest a pattern that will apply to larger words. Let's try a few larger cases:

```
(define (explode1 wd)
  (se wd))

(define (explode2 wd)
  (se (first wd) (last wd)))

(define (explode3 wd)
  (se (first wd) (first (bf wd)) (last wd)))
```

With `explode3` the procedure is starting to get complicated enough that we want to find a way to use `explode2` to help. What `explode3` does is to pull three separate letters out of its argument word, and collect the three letters in a sentence. Here's a sample:

```
> (explode3 'tnt)
(T N T)
```

`Explode2` pulls *two* letters out of a word and collects them in a sentence. So we could let `explode2` deal with two of the letters of our three-letter argument, and handle the remaining letter separately:

```
(define (explode3 wd)
  (se (first wd) (explode2 (bf wd))))
```

We can use similar reasoning to define `explode4` in terms of `explode3`:

```
(define (explode4 wd)
  (se (first wd) (explode3 (bf wd))))
```

Now that we see the pattern, what's the base case? Our first three numbered `explode`s are all different in shape from `explode3`, but now that we know what the

pattern should be we'll find that we can write `explode2` in terms of `explode1`, and even `explode1` in terms of `explode0`:

```
(define (explode2 wd)
  (se (first wd) (explode1 (bf wd))))

(define (explode1 wd)
  (se (first wd) (explode0 (bf wd))))
```

We would never have thought to write `explode1` in that roundabout way, especially since `explode0` pays no attention to its argument, so computing the `butfirst` doesn't contribute anything to the result, but by following the pattern we can let the recursive case handle one-letter and two-letter words, so that only zero-letter words have to be special:

```
(define (explode wd)
  (if (empty? wd)
      '()
      (se (first wd) (explode (bf wd)))))
```

Now for `letter-pairs`. What's the smallest word we can use as its argument? Empty and one-letter words have no letter pairs in them:

```
(define (letter-pairs0 wd)
  '())

(define (letter-pairs1 wd)
  '())
```

This pattern is not very helpful.

```
(define (letter-pairs2 wd)
  (se wd))

(define (letter-pairs3 wd)
  (se (bl wd) (bf wd)))

(define (letter-pairs4 wd)
  (se (bl (bl wd))
      (bl (bf wd))
      (bf (bf wd))))
```

Again, we want to simplify `letter-pairs4` by using `letter-pairs3` to help. The problem is similar to `explode`: The value returned by `letter-pairs4` is a three-word sentence, and we can use `letter-pairs3` to generate two of those words.

```
> (letter-pairs4 'nems)
(NE  EM MS )
```

This gives rise to the following procedure:

```
(define (letter-pairs4 wd)
  (se (bl (bl wd))
      (letter-pairs3 (bf wd))))
```

Does this pattern work for defining `letter-pairs5` in terms of `letter-pairs4`?

```
(define (letter-pairs5 wd)                    ;; wrong
  (se (bl (bl wd))
      (letter-pairs4 (bf wd))))

> (letter-pairs5 'bagel)
(BAG AG GE EL)
```

The problem is that `(bl (bl wd))` means "the first two letters of `wd`" only when `wd` has four letters. In order to be able to generalize the pattern, we need a way to ask for the first two letters of a word that works no matter how long the word is. You wrote a procedure to solve this problem in Exercise 5.15:

```
(define (first-two wd)
  (word (first wd) (first (bf wd))))
```

Now we can use this for `letter-pairs4` and `letter-pairs5`:

```
(define (letter-pairs4 wd)
  (se (first-two wd) (letter-pairs3 (bf wd))))

(define (letter-pairs5 wd)
  (se (first-two wd) (letter-pairs4 (bf wd))))
```

This pattern *does* generalize.

```
(define (letter-pairs wd)
  (if (<= (count wd) 1)
      '()
      (se (first-two wd)
          (letter-pairs (bf wd)))))
```

Note that we treat two-letter and three-letter words as recursive cases and not as base cases. Just as in the example of `explode`, we noticed that we could rewrite `letter-pairs2` and `letter-pairs3` to follow the same pattern as the larger cases:

```
(define (letter-pairs2 wd)
  (se (first-two wd)
      (letter-pairs1 (bf wd))))

(define (letter-pairs3 wd)
  (se (first-two wd)
      (letter-pairs2 (bf wd))))
```

Pitfalls

⇒ Every recursive procedure must include two parts: one or more recursive cases, in which the recursion reduces the size of the problem, and one or more base cases, in which the result is computable without recursion. For example, our first attempt at `downup` fell into this pitfall because we had no base case.

⇒ Don't be too eager to write the recursive procedure. As we showed in the `letter-pairs` example, what looks like a generalizable pattern may not be.

Boring Exercises

11.1 Write `downup4` using only the word and sentence primitive procedures.

11.2 [8.12]* When you teach a class, people will get distracted if you say "um" too many times. Write a `count-ums` that counts the number of times "um" appears in a sentence:

* Exercise 8.12 in Part III asks you to solve this same problem using higher-order functions. Here we are asking you to use recursion. Whenever we pose the same problem in both parts, we'll cross-reference them in brackets as we did here. When you see the problem for the second time, you might want to consult your first solution for ideas.

```
> (count-ums
   '(today um we are going to um talk about the combining um method))
3
```

Here are some special-case `count-ums` procedures for sentences of particular lengths:

```
(define (count-ums0 sent)
  0)

(define (count-ums1 sent)
  (if (equal? 'um (first sent))
      1
      0))

(define (count-ums2 sent)
  (if (equal? 'um (first sent))
      (+ 1 (count-ums1 (bf sent)))
      (count-ums1 (bf sent))))

(define (count-ums3 sent)
  (if (equal? 'um (first sent))
      (+ 1 (count-ums2 (bf sent)))
      (count-ums2 (bf sent))))
```

Write `count-ums` recursively.

11.3 [8.13] Write a procedure `phone-unspell` that takes a spelled version of a phone number, such as `POPCORN`, and returns the real phone number, in this case `7672676`. You will need a helper procedure that translates a single letter into a digit:

```
(define (unspell-letter letter)
  (cond ((member? letter 'abc) 2)
        ((member? letter 'def) 3)
        ((member? letter 'ghi) 4)
        ((member? letter 'jkl) 5)
        ((member? letter 'mno) 6)
        ((member? letter 'prs) 7)
        ((member? letter 'tuv) 8)
        ((member? letter 'wxy) 9)
        (else 0)))
```

Here are some some special-case `phone-unspell` procedures:

```
(define (phone-unspell1 wd)
  (unspell-letter wd))

(define (phone-unspell2 wd)
  (word (unspell-letter (first wd))
        (unspell-letter (first (bf wd)))))

(define (phone-unspell3 wd)
  (word (unspell-letter (first wd))
        (unspell-letter (first (bf wd)))
        (unspell-letter (first (bf (bf wd))))))
```

Write `phone-unspell` recursively.

Real Exercises

Use recursion to solve these problems, not higher order functions (Chapter 8)!

11.4 Who first said "use what you have to get what you need"?

11.5 Write a procedure `initials` that takes a sentence as its argument and returns a sentence of the first letters in each of the sentence's words:

```
> (initials '(if i needed someone))
(I I N S)
```

11.6 Write a procedure `countdown` that works like this:

```
> (countdown 10)
(10 9 8 7 6 5 4 3 2 1 BLASTOFF!)

> (countdown 3)
(3 2 1 BLASTOFF!)
```

11.7 Write a procedure `copies` that takes a number and a word as arguments and returns a sentence containing that many copies of the given word:

```
> (copies 8 'spam)
(SPAM SPAM SPAM SPAM SPAM SPAM SPAM SPAM)
```